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Non-factorisable $\mathbb{Z}_2 \times \mathbb{Z}_2$ heterotic orbifold models and Yukawa couplings

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Abstract: We classify compactification lattices for supersymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds. These lattices include factorisable as well as non-factorisable six-tori. Different models lead to different numbers of fixed points/tori. A lower bound on the number of fixed tori per twisted sector is given by four, whereas an upper bound consists of 16 fixed tori per twisted sector. Thus, these models have a variety of generation numbers. For example, in the standard embedding, the smallest number of net generations among these classes of models is equal to six, while the largest number is 48. Conditions for allowed Wilson lines and Yukawa couplings are derived.

Keywords: Superstrings and Heterotic Strings, Superstring Vacua.

Contents

1. Introduction

The Theory of Superstrings successfully unifies the concepts of Quantum Field Theory and General Relativity. In order to claim that it incorporates a unification of all forces observed in nature, one has to prove the existence of string models reproducing standard particle physics. The best way to prove that existence consists in the construction of explicit models since that allows also to investigate phenomenological implications of string theory. The, perhaps most traditional, attempt of identifying realistic string models is given by heterotic orbifold constructions [1, 2]. In more recent years, this line of research was boosted by the observation that phenomenological properties can be connected to geometrical properties of the orbifold [3–5]. Examples for quantities which are directly tied to geometry are the Kähler potential for twisted sector states as well as Yukawa couplings [6–10]. (For interesting applications see e.g. ref. [11].)

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In a parallel development, semi realistic models have been obtained in the free fermionic formulation of heterotic strings [12, 13]. Although there are some indications [14] that these models are related to $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds a precise connection has not been worked out in general. Hence a geometric picture is missing for many free fermionic models.

Most of the earlier explicit $\mathbb{Z}_2 \times \mathbb{Z}_2$ heterotic constructions are based on factorisable compactification lattices [15, 4] (for different compactifications for this and other orbifolds see, however, [1, 16 − 18]). In the current paper, we are going to further explore the approach taken in [19] where the explicit construction on $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds of non factorisable six-tori with and without Wilson lines is presented. These types of orbifolds are among the simplest phenomenologically interesting constructions of string theory models. Since it is believed, that string theory allows for a large number of realistic models, simplicity provides a good principle to impose. Moreover, the phenomenologically attractive features of the previously mentioned free fermionic constructions and their conjectured relation to $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds suggest a potential relevance of $\mathbb{Z}_2 \times \mathbb{Z}_2$ for real particle physics.

Replacing factorisable compactification lattices by non factorisable ones affects the fixed point structure of the internal space. In $\mathbb{Z}_2 \times \mathbb{Z}_2$ chiral matter originates from twisted sector states and, hence, depends on the fixed point structure. But also other relevant aspects like the selection rules [6, 20] for Yukawa couplings are affected. On the factorisable $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, only 'diagonal' Yukawa couplings are allowed. Here, the phrase "diagonal Yukawa couplings" means that when we choose two fields, the other field to be allowed to couple is uniquely fixed. Furthermore, allowed Yukawa couplings are of $O(1)$. The situation will change on non-factorisable $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models.

In the present paper, we will construct eight classes of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds. All of these classes lead to $N = 1$ supersymmetry in four dimensions. Topological properties of the orbifold differ from class to class but do not change for models within the same class.

The paper is organised as follows. In the next section we discuss some $T^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds, and specify under which circumstances two orbifolds can be considered as equivalent. In section 3 we identify building blocks for the classification of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds. Section 4 describes how to compute the number of families and anti-families from modular invariance. In section 5 we discuss consistency conditions on discrete Wilson lines and selection rules for Yukawa couplings. We summarise and discuss our results in section 6. Appendix A provides the details for eight classes of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds.

2. $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds of T^3

As a warm up, we are going to discuss orbifolds of T^3. These will serve as easy examples to identify equivalent orbifold compactifications and also appear as substructures of the relevant orbifolds of T^6 to be studied afterwards. We will call orbifolds equivalent if they are connected by a continuous deformation of geometric moduli. From a phenomenological perspective, we would call two orbifolds equivalent if the massless spectrum and the selection rules for allowed couplings are the same. The latter understanding of equivalence follows from our definition. In the following, we present two pairs of equivalent orbifolds of T^3, and address both of the above aspects of equivalence.

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2.1 SU(4)/$\mathbb{Z}_2 \times \mathbb{Z}_2$ versus SU(3)×SU(2)/$\mathbb{Z}_2 \times \mathbb{Z}_2$

We specify the orbifold action on a Cartesian coordinate system of a three dimensional Euclidean space by associating the generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$, θ_1 and θ_2, with three by three transformation matrices. In the present example, we choose

$$\theta_1 = \mathrm{diag}\,(-1,1,1) \quad , \quad \theta_2 = -1\!\!1_3. \tag{2.1}$$

The third non trivial element is given by the product $\theta_3 = \theta_1\theta_2$. The three dimensional space is compactified by identifying points differing by vectors of a three dimensional lattice. We take the SU(4) root lattice $\Lambda_{SU(4)}$ with the simple roots

$$\alpha_1 = \left(\sqrt{2},0,0\right) \quad , \quad \alpha_2 = \left(-\frac{\sqrt{2}}{2},\sqrt{\frac{3}{2}},0\right) \quad , \quad \alpha_3 = \left(0,-\sqrt{\frac{2}{3}},\frac{2}{\sqrt{3}}\right). \tag{2.2}$$

On the lattice vectors, θ_1 acts as

$$\theta_1\alpha_1 = -\alpha_1, \qquad \theta_1\alpha_2 = \alpha_1+\alpha_2, \qquad \theta_1\alpha_3 = \alpha_3, \tag{2.3}$$

which is the Weyl reflection at α_1.

Next, we count the number of points/tori fixed under the action of the non trivial orbifold elements θ_1, θ_2 and θ_3. We will employ a Lefshets fixed point theorem in order to compute them. The element θ_1 acts trivially on two directions and hence leaves tori fixed. For the computation two lattices are relevant.

The sublattice $(1-\theta_1)\Lambda_{SU(4)}$ is spanned by α_1. In the non-factorisable orbifold, the lattice, which is normal to the lattice invariant under the twist θ, is non-trivial and important. We denote it by N_θ. The normal lattice N_{θ_1} is spanned by α_1. The number of independent fixed points/tori is obtained as [21] (see also [22, 19])

$$\frac{Vol\left((1-\theta_1)\Lambda_{SU(4)}\right)}{Vol\left(N_{\theta_1}\right)}, \tag{2.4}$$

that is, the number of lattice sites on N_{θ_1}, which are not identified up to $(1-\theta_1)\Lambda_{SU(4)}$. The T_1 sector[1] has a single fixed torus corresponding to[2] $(\theta_1, 0)$. For the T_2 sector, the sublattice $(1-\theta_2)\Lambda_{SU(4)}$ is spanned by $2\alpha_i$ ($i=1,2,3$), and there are eight fixed points corresponding to $(\theta_2, \sum m_i^{(2)}\alpha_i)$ with $m_i^{(2)} = 0,1$ for $i = 1,2,3$. For the T_3 sector, the sublattice $(1-\theta_3)\Lambda_{SU(4)}$ is spanned by $(\alpha_1+2\alpha_2)$ and $2\alpha_3$, and the normal lattice N_{θ_3} is spanned by $(\alpha_1+2\alpha_2)$ and α_3. Thus, the T_3 sector has two independent fixed tori, $(\theta_3, m_3^{(3)}\alpha_3)$ with $m_3^{(3)} = 0,1$.

Now, let us present the orbifold of another T^3 which has the same topology as the one discussed, so far. We choose the same representation of the orbifold group on Cartesian

[1]The T_i sector corresponds to twisted strings closing only after points are identified by a θ_i action.

[2]Instead of giving the coordinates of the fixed torus we associate it to the space group element leaving it fixed also on the non compact three space. On the orbifold, this identifies the fixed torus uniquely.

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coordinates but replace the compactification lattice by the root lattice of SU(3)×SU(2), i.e. the lattice is generated by

$$\alpha_1 = \left(\sqrt{2}, 0, 0\right) \quad , \quad \alpha_2 = \left(-\frac{\sqrt{2}}{2}, \sqrt{\frac{3}{2}}, 0\right) \quad , \quad e_3 = \left(0, 0, \sqrt{2}\right) . \tag{2.5}$$

The action of θ_1 on the basis of the lattice is

$$\theta_1\alpha_1 = -\alpha_1, \qquad \theta_1\alpha_2 = \alpha_1 + \alpha_2, \qquad \theta_1 e_3 = e_3, \tag{2.6}$$

which is again a Weyl reflection at α_1.

The sublattice $(1-\theta_1)\Lambda_{SU(3)\times SU(2)}$ is spanned by α_1, and the normal lattice N_{θ_1} is spanned by α_1. Thus, the T_1 sector has a single independent fixed torus $(\theta_1, 0)$. The sublattice $(1-\theta_2)\Lambda_{SU(3)\times SU(2)}$ is spanned by $2\alpha_i$ $(i = 1, 2)$ and $2e_3$, and the T_2 sector has eight fixed points, $(\theta_2, \sum_i m_i^{(2)}\alpha_i + m_3^{(2)}e_3)$. The sublattice $(1-\theta_3)\Lambda_{SU(3)\times SU(2)}$ is spanned by $(\alpha_1 + 2\alpha_2)$ and $2e_3$, and the normal lattice N_{θ_3} is spanned by $(\alpha_1 + 2\alpha_2)$ and e_3. There are two independent fixed tori $(\theta_3, m_3^{(3)}e_3)$ with $m_3^{(3)} = 0, 1$.

Now, let us compare the above two types of orbifolds. Obviously, they have the same number of fixed points. Moreover, the structure of $(1-\theta)\Lambda$ and N_θ is the same. That leads to the same selection rule of allowed couplings. For the calculation, it is simpler to use a direct product of smaller mutually orthogonal lattices.

The above consideration indicates that, topologically, the two orbifolds might be equivalent. In the following, we will see that they can indeed be connected by continuous deformations of geometric moduli. In Cartesian coordinates the geometric moduli of our orbifolds are the following metric components

$$G_{11}, \;\; G_{22}, \;\; G_{23}, \;\; G_{33}. \tag{2.7}$$

Turning on these moduli changes the scalar products of lattice vectors. For the system (2.2) one obtains

$$\alpha_i \cdot \alpha_j = \begin{pmatrix} 2G_{11} & -G_{11} & 0 \\ -G_{11} & \frac{1}{2}G_{11} + \frac{3}{2}G_{22} & \sqrt{2}G_{23} - G_{22} \\ 0 & \sqrt{2}G_{23} - G_{22} & \frac{2}{3}G_{22} + \frac{4}{3}G_{33} - \frac{4\sqrt{2}}{3}G_{23} \end{pmatrix} . \tag{2.8}$$

For the Euclidean metric one obtains the Cartan matrix of SU(4), whereas one can find also a metric yielding the Cartan matrix of SU(3)×SU(2)

$$\alpha_i \cdot \alpha_j = \begin{cases} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} & \text{for } G_{ij} = \delta_{ij} \\[2ex] \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} & \text{for } \begin{array}{l} G_{11} = G_{22} = \sqrt{2}G_{23} = 1, \\ G_{33} = 2. \end{array} \end{cases} \tag{2.9}$$

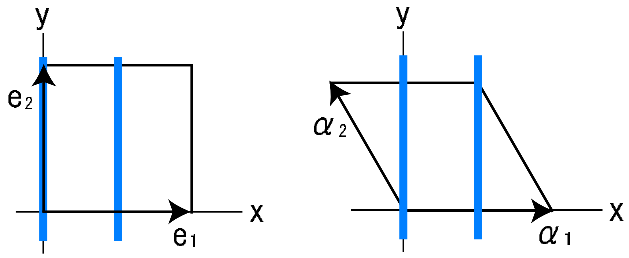

Figure 1: Fixed lines on the SU(2)×SU(2)/$\mathbb{Z}_2 \times \mathbb{Z}_2$ and the SU(3)/$\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds.

The second metric in (2.9) can be transformed into a Euclidean metric by coordinate changes. Since these leave the scalar products invariant, one obtains an SU(3)×SU(2) root lattice embedded in Euclidean space.

There is a, perhaps easier, way to see that the two orbifolds are connected. Instead of varying the metric one could change the basis in the 2-3 plane whilst keeping the metric fixed. Replacing α_3 in (2.2) by e_3 in (2.6) can be easily achieved by such a change of basis.

Finally, using the SU(3)× SU(2)/$\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, we give an intuitive explanation to show how the number of fixed points/tori is reduced on non-factorisable orbifolds. Let us concetrate its non-trivial part, i.e. SU(3)/$\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, and compare with the factorisable orbifold, i.e. the SU(2) ×SU(2)/$\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. The SU(3) lattice is spanned by α_1 and α_2, while the SU(2) ×SU(2) lattice is spanned by e_1 and e_2. Then, we consider the twist θ_1, whose action on the SU(3) lattice is the same as eq. (2.6), i.e. $\theta_1\alpha_1 = -\alpha_1$ and $\theta_1\alpha_2 = \alpha_1 + \alpha_2$. On the other hand, the twist θ_1 acts on the SU(2)×SU(2) lattice as $\theta_1 e_1 = -e_1$ and $\theta_1 e_2 = e_2$. Under the twist θ_1, there are two fixed lines on the SU(2)×SU(2) torus, which are shown in the left chart of figure 1. They correspond to coordinates ye_2 and $e_1/2 + ye_2$ with a continuous parameter y, and are represented by space group elements, $(\theta_1, 0)$ and (θ_1, e_1). Similarly, there are two fixed lines under θ_1 on the SU(3) torus as shown in the right chart of figure 1. However, these fixed lines must be identified on the SU(3) torus, because both of them pass through lattice sites. In other words, a point on one fixed line can be mapped to a point on the other fixed line with a shift by $\alpha_1 + \alpha_2$. This fact is also understood in terms of space group, that is, the sublattice $(1-\theta_1)\Lambda_{\mathrm{SU}(3)}$ is spanned by α_1. Note that the sublattice $(1-\theta_1)\Lambda_{\mathrm{SU}(2)}$ is spanned by $2e_1$. Thus, we have a single independent fixed line under θ_1 on the SU(3) torus, while there are two independent fixed lines on the SU(2) ×SU(2) torus.

2.2 SO(6)/$\mathbb{Z}_2 \times \mathbb{Z}_2$ versus SO(4)×SU(2)/$\mathbb{Z}_2 \times \mathbb{Z}_2$

In the present example we represent the orbifold action on Cartesian coordinates by

$$\theta_1 = \mathrm{diag}\,(1, 1, -1) \quad , \quad \theta_2 = -\mathbb{1}_3. \tag{2.10}$$

As a compactification lattice we choose the SO(6) lattice with simple roots

$$\alpha_1 = (0, 1, -1) \quad , \quad \alpha_2 = (1, -1, 0) \quad , \quad \alpha_3 = (0, 1, 1)\,. \tag{2.11}$$

With SU(4) being the universal covering of SO(6) this lattice is equivalent to the previously considered lattice (2.2). Our terminology is inspired by the observation that the way roots

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have been written fits into a general SU(N) respectively SO(2N) pattern. In the present context, the real difference is that now θ_1 acts as an outer automorphism of the Dynkin diagram

$$\theta_1\alpha_1 = \alpha_3, \qquad \theta_1\alpha_2 = \alpha_2, \qquad \theta_1\alpha_3 = \alpha_1. \tag{2.12}$$

The sublattices $(1-\theta_i)\Lambda_{SU(4)}$ and the normal lattices N_{θ_i} are spanned by

$$\begin{array}{ll} (1-\theta_1)\Lambda_{SU(4)} : \{\alpha_1-\alpha_3\}, & N_{\theta_1} : \{\alpha_1-\alpha_3\}, \\ (1-\theta_2)\Lambda_{SU(4)} : \{2\alpha_1, 2\alpha_2, 2\alpha_3\}, & N_{\theta_2} : \{\alpha_1, \alpha_2, \alpha_3\}, \\ (1-\theta_3)\Lambda_{SU(4)} : \{(\alpha_1+\alpha_3), 2\alpha_2\}, & N_{\theta_3} : \{(\alpha_1+\alpha_3), \alpha_2\}. \end{array} \tag{2.13}$$

The T_1, T_2 and T_3 sectors have one, eight and two fixed points, respectively, which are denoted as

$$(\theta_1, 0), \qquad (\theta_2, \sum_{i=1}^{3} m_i^{(2)}\alpha_i), \qquad (\theta_3, m_2^{(3)}\alpha_2), \tag{2.14}$$

where $m_i^{(2)} = 0,1$ and $m_2^{(3)} = 0,1$.

Next, we are going to compare this to a compactification on an SO(4)×SU(2) lattice with the same action of the orbifold group on Cartesian coordinates. The compactification lattice is (again the terminology of writing SO(4) instead of SU(2)^2 is inspired by the way roots α_1 and α_3 are written)

$$\alpha_1 = (0,1,-1) \ \ , \ \ e_2 = \left(\sqrt{2},0,0\right) \ \ , \ \ \alpha_3 = (0,1,1)\,. \tag{2.15}$$

The action on these lattice vectors is the same as in (2.12) with α_2 replaced by e_2.

The sublattices $(1-\theta_i)\Lambda_{SO(4)\times SU(2)}$ and the normal lattices N_{θ_i} are spanned by

$$\begin{array}{ll} (1-\theta_1)\Lambda_{SO(4)\times SU(2)} : \{\alpha_1-\alpha_3\}, & N_{\theta_1} : \{\alpha_1-\alpha_3\}, \\ (1-\theta_2)\Lambda_{SO(4)\times SU(2)} : \{2\alpha_1, 2e_2, 2\alpha_3\}, & N_{\theta_2} : \{\alpha_1, e_2, \alpha_3\}, \\ (1-\theta_3)\Lambda_{SO(4)\times SU(2)} : \{(\alpha_1+\alpha_3), 2e_2\}, & N_{\theta_3} : \{(\alpha_1+\alpha_3), e_2\}. \end{array} \tag{2.16}$$

The T_1, T_2 and T_3 sectors have one, eight and two fixed points, respectively, which are denoted as

$$(\theta_1, 0), \qquad (\theta_2, \sum_{i=1,3} m_i^{(2)}\alpha_i + m_2^{(2)}e_2), \qquad (\theta_3, m_2^{(3)}e_2), \tag{2.17}$$

where $m_i^{(2)} = 0,1$ and $m_2^{(3)} = 0,1$.

The last two orbifolds lead to the same number of fixed points. Furthermore, the structure of $(1-\theta)\Lambda$ and N_θ is the same. Thus, selection rules for allowed couplings impose the same conditions for the two orbifolds.

Finally, we argue that the two orbifolds, SO(6)/$\mathbb{Z}_2 \times \mathbb{Z}_2$ and SO(4)×SU(2)/$\mathbb{Z}_2 \times \mathbb{Z}_2$, can be continuously deformed into each other by changing geometric moduli. The orbifold action (2.10) leaves the metric components

$$G_{11}, \ G_{12}, \ G_{22}, \ G_{33} \tag{2.18}$$

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invariant. In such general background the scalar product of the simple SO(6) roots (2.11) reeds

$$\alpha_i \cdot \alpha_j = \begin{pmatrix} G_{22}+G_{33} & G_{12}-G_{22} & G_{22}-G_{33} \\ G_{12}-G_{22} & G_{11}-2G_{12}+G_{22} & G_{12}-G_{22} \\ G_{22}-G_{33} & G_{12}-G_{22} & G_{22}+G_{33} \end{pmatrix}. \tag{2.19}$$

There are special points in moduli space where the above becomes the Cartan matrix of SO(6) or SO(4)×SU(2),

$$\alpha_i \cdot \alpha_j = \begin{cases} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \text{ for } G_{ij} = \delta_{ij} \\ \\ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{for } \begin{array}{l} G_{12} = G_{22} = G_{33} = 1, \\ G_{11} = 3. \end{array} \end{cases} \tag{2.20}$$

Alternatively, one can deform geometric moduli by changing the basis in the 1-2 plane while keeping the metric Euclidean. Such a basis change can be employed to replace α_2 in (2.11) by e_2 in (2.15).

Let us mention a subtlety which will be of some importance later. Instead of (2.11) we could have chosen the basis

$$\tilde{\alpha}_1 = (-1,1,0) \ \ ,\tilde{\alpha}_2 = (0,-1,1) \ \ , \ \ \tilde{\alpha}_3 = (1,1,0)\,, \tag{2.21}$$

since this basis generates the same lattice. Now, the projection on the 1-2 plane yields vectors $(-1,0)$, $(0,-1)$ and $(1,1)$ and there is no non degenerate coordinate transformation such that one of these vectors becomes orthogonal to the two others. So, our previous argument for the equivalence to an SO(4)×SU(2) lattice seems to fail. However, the systems (2.11) and (2.21) are related by a coordinate transformation

$$\tilde{x}^1 = x^3 \ \ , \ \ \tilde{x}^2 = x^2 \ \ , \ \ \tilde{x}^3 = x^1. \tag{2.22}$$

The metric component G_{13} is projected out by the orbifold. The coordinate transformation (2.22) does not generate such a component from a vanishing one and hence does not induce a projected metric deformation. (Indeed, the Euclidean metric is invariant under (2.22).) So, we can safely go back to the choice (2.11) and perform the previous deformations to connect to the SO(4)×SU(2) lattices. The conclusion from this short discussion is that the basis of a lattice can be replaced by another basis of the same lattice. This should be always the case as long as the orbifold acts as a lattice automorphism. (This statement seems somewhat trivial. We mentioned it, even so, since it will be important for later applications.)

In summary, our examples illustrate that whenever the orbifold action is such that in a Cartesian basis there are off diagonal metric moduli, the compactification lattice can be continuously deformed into a lattice where one of the basis vectors is orthogonal to the rest of the lattice. This is the case when the θ_1 as well as the θ_2 action on a two dimensional subspace of $\mathbb{R}^3$ equals $\pm 1\!\!1_2$.

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3. $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds of T^6

Our ultimate goal is it to construct phenomenologically interesting string models. A prominent route to take is to compactify heterotic string theory on a six dimensional orbifold. Such models easily lead to $N = 1$ supersymmetric theories with non Abelian gauge groups and chiral matter in four dimensions. The requirement of having exactly $N = 1$ unbroken supersymmetry in four dimensions imposes constraints on the orbifold action. In the present paper, we are interested in one of the simplest orbifold groups *viz.* $\mathbb{Z}_2 \times \mathbb{Z}_2$. In order to obtain $N = 1$ supersymmetry in four dimensions we have to specify the action of the two $\mathbb{Z}_2 \times \mathbb{Z}_2$ generators, θ_1 and θ_2, on a Cartesian coordinate system as follows

$$\theta_1 = \mathrm{diag}\,(-1,-1,-1,-1,1,1) \quad , \quad \theta_2 = \mathrm{diag}\,(1,1,-1,-1,-1,-1)\,. \tag{3.1}$$

This choice is unique up to relabelling the coordinates.

Next, we would like to classify all possible compactification lattices. From the action of the orbifold elements (3.1) and the conclusions of the previous section we see that whenever a lattice extends over more than three directions we can continuously deform it to a lattice consisting of two orthogonal factors. Therefore, any six dimensional lattice can be deformed into a product of mutually orthogonal three dimensional lattices and we are left with the task of classifying three dimensional lattices.

Before doing so, let us illustrate the above statement at the example of an SO(12) compactification lattice. The basis vectors of the compactification lattice are the SO(12) simple roots

$$\begin{aligned}
\alpha_1 &= (1,-1,0,0,0,0)\,,\\
\alpha_2 &= (0,1,-1,0,0,0)\,,\\
\alpha_3 &= (0,0,1,-1,0,0)\,,\\
\alpha_4 &= (0,0,0,1,-1,0)\,,\\
\alpha_5 &= (0,0,0,0,1,-1)\,,\\
\alpha_6 &= (0,0,0,0,1,1)\,.
\end{aligned} \tag{3.2}$$

The geometric moduli correspond to changes of the coordinate basis in the 1-2, 3-4 and 5-6 plane while keeping the metric fixed, and, as discussed in the end of the previous section, changing the basis of a lattice. Changing $x^2 \to x^2 + x^1$ we can 'decouple' α_1, and (after rescaling $x^1 \to \sqrt{2}x^1$) obtain an SU(2)×SO(10) lattice. For the SO(10) lattice we replace the basis $\alpha_2, \ldots, \alpha_6$ by an equivalent one and obtain

$$\begin{aligned}
e_1 &= \left(\sqrt{2},0,0,0,0,0\right)\,,\\
\tilde{\alpha}_2 &= (0,1,-1,0,0,0)\,,\\
\tilde{\alpha}_3 &= (0,1,1,0,0,0)\,,\\
\tilde{\alpha}_4 &= (0,-1,0,1,0,0)\,,\\
\tilde{\alpha}_5 &= (0,0,0,-1,1,0)\,,\\
\tilde{\alpha}_6 &= (0,0,0,0,-1,1)\,.
\end{aligned} \tag{3.3}$$

Now, we perform another allowed metric deformation by replacing $x^5 \to x^5 + x^6$ and $x^6 \to \sqrt{2}x^6$. Then $\tilde{\alpha}_2, \ldots \tilde{\alpha}_5$ generate an SO(8) root lattice whereas $\tilde{\alpha}_6$ generates another orthogonal SU(2) lattice. For the SO(8) lattice we pick again an equivalent set of basis vectors

$$\begin{aligned}
e_1 &= \left(\sqrt{2}, 0, 0, 0, 0, 0\right), \\
\hat{\alpha}_2 &= (0, 1, 0, 0, -1, 0), \\
\hat{\alpha}_3 &= (0, 1, 0, 0, 1, 0), \\
\hat{\alpha}_4 &= (0, -1, 1, 0, 0, 0), \\
\hat{\alpha}_5 &= (0, 0, -1, 1, 0, 0), \\
e_6 &= \left(0, 0, 0, 0, 0, \sqrt{2}\right).
\end{aligned} \tag{3.4}$$

Finally, we deform geometric moduli in the 3-4 plane by replacing $x^3 \to x^3 + x^4$ and $x^4 \to \sqrt{2}x^4$ we obtain as a compactification lattice the root lattice of[3] SU(4)×SU(2)^3

$$\begin{aligned}
e_1 &= \left(\sqrt{2}, 0, 0, 0, 0, 0\right), \\
\hat{\alpha}_2 &= (0, 1, 0, 0, -1, 0), \\
\hat{\alpha}_3 &= (0, 1, 0, 0, 1, 0), \\
\hat{\alpha}_4 &= (0, -1, 1, 0, 0, 0), \\
e_5 &= \left(0, 0, 0, \sqrt{2}, 0, 0\right), \\
e_6 &= \left(0, 0, 0, 0, 0, \sqrt{2}\right).
\end{aligned} \tag{3.5}$$

The SU(4) lattice, generated by $\hat{\alpha}_2$, $\hat{\alpha}_3$, $\hat{\alpha}_4$, is extended along the 2, 3 and 5 direction, and there are not enough geometric moduli to decompose it further (G_{23}, G_{25} and G_{35} are projected out by the orbifold action). The SO(12) example illustrates our previous statement that three dimensional factorisable and non-factorisable lattices appear as building blocks for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds when the orbifold action is given by (3.1) such that $N = 1$ supersymmetry is unbroken. In terms of non-factorisable lattices the building blocks are thus given by three, two and one dimensional lattices. We are going to classify these in the following.

A non-factorisable three dimensional lattice cannot be decomposed further if the orbifold action on the three dimensional subspace is given by (up to permuting θ_1, θ_2 and $\theta_3 = \theta_1\theta_2$)

$$\theta_1 = (-1, -1, 1), \qquad \theta_2 = (1, -1, -1), \qquad \theta_3 = (-1, 1, -1). \tag{3.6}$$

We use the basis of simple roots (2.11), and twists θ_i (3.6). (We obtain the same result when we use the basis (2.2).)

The twists θ_i are represented by Weyl reflections and outer automorphism. For example, θ_1 is a product of the outer automorphism swapping α_1 with α_3 and a total $\mathbb{Z}_2$ rotation, $\alpha_i \to -\alpha_i$, while the twist θ_2 is a sum of two Weyl reflections at α_1 and α_3. Thus,

[3] From now on we drop our previous differentiation of SO(6) and SU(4) lattices.

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the twists θ_i transform the SU(4) simple roots as

$$\begin{aligned}
&\theta_1\alpha_1 = -\alpha_3, \qquad \theta_1\alpha_2 = -\alpha_2, \qquad \theta_1\alpha_3 = -\alpha_1, \\
&\theta_2\alpha_1 = -\alpha_1, \qquad \theta_2\alpha_2 = \sum_{i=1}^{3}\alpha_i, \qquad \theta_2\alpha_3 = -\alpha_3, \\
&\theta_3\alpha_1 = \alpha_3, \qquad \theta_3\alpha_2 = -\sum_{i=1}^{3}\alpha_i, \qquad \theta_3\alpha_3 = \alpha_1.
\end{aligned} \tag{3.7}$$

The sublattices $(1-\theta_i)\Lambda_{SU(4)}$ and the normal lattices N_{θ_i} are spanned by

$$\begin{array}{ll}
(1-\theta_1)\Lambda_{SU(4)} : \{(\alpha_1+\alpha_3), 2\alpha_2\}, & N_{\theta_1} : \{(\alpha_1+\alpha_3), \alpha_2\}, \\
(1-\theta_2)\Lambda_{SU(4)} : \{(\alpha_1+\alpha_3), 2\alpha_1\}, & N_{\theta_2} : \{\alpha_1, \alpha_3\}, \\
(1-\theta_3)\Lambda_{SU(4)} : \{2(\alpha_2+\alpha_3), (\alpha_1-\alpha_3)\}, & N_{\theta_3} : \{(\alpha_2+\alpha_3), (\alpha_1-\alpha_3)\}.
\end{array} \tag{3.8}$$

Thus, each of the T_1, T_2 and T_3 sectors contains two fixed tori,

$$T_1 : (\theta_1, m^{(1)}\alpha_2), \qquad T_2 : (\theta_2, m^{(2)}\alpha_1), \qquad T_3 : (\theta_3, m^{(3)}(\alpha_1+\alpha_2)), \tag{3.9}$$

where $m^{(a)} = 0, 1$ for $a = 1, 2, 3$.

A non-factorisable two dimensional lattice cannot be decomposed further by deforming geometric moduli if the orbifold acts on the two dimensional subspace as

$$\theta_1 = \mathrm{diag}\,(-1, 1) \ \ , \ \ \theta = -\mathbb{1}_2. \tag{3.10}$$

First, we take the following SU(3) lattice as a compactification lattice:

$$\alpha_1 = \left(\sqrt{2}, 0\right) \ \ , \ \ \alpha_2 = \left(-\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right). \tag{3.11}$$

On this root lattice θ_1 acts as a Weyl reflection at α_1, and θ_2 as multiplying all roots with a minus sign

$$\begin{aligned}
&\theta_1\alpha_1 = -\alpha_1, \qquad \theta_1\alpha_2 = \alpha_1+\alpha_2, \\
&\theta_2\alpha_1 = -\alpha_1, \qquad \theta_2\alpha_2 = -\alpha_2, \\
&\theta_3\alpha_1 = \alpha_1, \qquad \theta_3\alpha_2 = -\alpha_1-\alpha_2.
\end{aligned} \tag{3.12}$$

The sublattices $(1-\theta_i)\Lambda_{SU(3)}$ and the normal lattices N_{θ_i} are obtained as

$$\begin{array}{ll}
(1-\theta_1)\Lambda_{SU(3)} : \{\alpha_1\}, & N_{\theta_1} : \{\alpha_1\}, \\
(1-\theta_2)\Lambda_{SU(3)} : \{2\alpha_1, 2\alpha_2\}, & N_{\theta_2} : \{\alpha_1, \alpha_2\}, \\
(1-\theta_3)\Lambda_{SU(3)} : \{\alpha_1+2\alpha_2\}, & N_{\theta_3} : \{\alpha_1+2\alpha_2\}.
\end{array} \tag{3.13}$$

The T_1, T_2 and T_3 sectors have one, four and one fixed tori,

$$T_1 : \ (\theta_1, 0), \qquad T_2 : (\theta_2, m_1^{(2)}\alpha_1 + m_2^{(2)}\alpha_2), \qquad T_3 : (\theta_3, 0), \tag{3.14}$$

where $m_1^{(2)}, m_2^{(2)} = 0, 1$.

Now, we are going to argue that the SU(3) lattice (3.11) can be deformed into an SO(4) lattice. At first sight this is not obvious since the only allowed deformations are separate rescalings of the two directions. However, the same SU(3) root lattice is generated by the alternative basis

$$\alpha_1 = \left(\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right) \quad , \quad \alpha_2 = \left(\frac{1}{\sqrt{2}}, -\sqrt{\frac{3}{2}}\right). \tag{3.15}$$

Now, θ_1 acts as an outer automorphism on the simple roots

$$\begin{aligned} \theta_1\alpha_1 &= \alpha_2, \qquad \theta_1\alpha_2 = \alpha_1, \\ \theta_2\alpha_1 &= -\alpha_1, \qquad \theta_2\alpha_2 = -\alpha_2, \\ \theta_3\alpha_1 &= -\alpha_2, \qquad \theta_3\alpha_2 = -\alpha_1. \end{aligned} \tag{3.16}$$

The system (3.15) can be geometrically deformed into an SO(4) root lattice. By rescaling the two directions one can bring it to the form

$$\alpha_1 = (1,1) \quad , \quad \alpha_2 = (1,-1). \tag{3.17}$$

These vectors span an SO(4) root lattice, which we distinguish from an $\mathrm{SU}(2)^2$ lattice by noticing that geometric moduli do not allow for separate changes in the length of the two roots. Below, we will use the SU(3) lattice (3.11) for the classification and not its equivalent versions (3.15) or (3.17).

In contrast to the SO(4) lattice (3.17), an SU(2)×SU(2) lattice will be given by

$$\alpha_1 = \left(\sqrt{2}, 0\right) \quad , \quad \alpha_2 = \left(0, \sqrt{2}\right). \tag{3.18}$$

This factorises into two one dimensional lattices. So, finally we consider a one dimensional lattice spanned by e_1. The three twists θ_i $(i = 1,2,3)$ are defined as

$$\theta_1 e_1 = -e_1, \qquad \theta_2 e_1 = -e_1, \qquad \theta_3 e_1 = e_1. \tag{3.19}$$

The sublattices $(1-\theta_i)\Lambda_{SU(2)}$ and the normal lattices N_{θ_i} are obtained as

$$\begin{aligned} (1-\theta_1)\Lambda_{SU(2)} &: \{2e_1\}, \qquad N_{\theta_1} : \{e_1\}, \\ (1-\theta_2)\Lambda_{SU(2)} &: \{2e_1\}, \qquad N_{\theta_2} : \{e_1\}. \end{aligned} \tag{3.20}$$

Both T_1 and T_2 sectors have two fixed tori,

$$T_1: \ (\theta_1, m^{(1)}e_1), \qquad T_2: (\theta_2, m_1^{(2)}e_1), \tag{3.21}$$

where $m^{(1)}, m^{(2)} = 0, 1$.

In summary, the building blocks of supersymmetric $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ are the three dimensional SU(4) root lattice, the two dimensional SU(3) root lattice (3.11) and the one dimensional SU(2) root lattice. Our results, including also the fixed point structure, are listed in table 1.

From these building blocks we can construct altogether eight classes of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds which are listed in the appendix.

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orbifold	α_i	θ_1	θ_2	θ_3	$T_1(\theta_1, V_1)$	$T_2(\theta_2, V_2)$	$T_3(\theta_3, V_3)$
$SU(4)$	α_1	$-\alpha_3$	$-\alpha_1$	α_3			
	α_2	$-\alpha_2$	$\sum_i^3 \alpha_i$	$-\sum_i^3 \alpha_i$	$m^{(1)}\alpha_2$	$m^{(2)}\alpha_1$	$m^{(3)}(\alpha_1 + \alpha_2)$
	α_3	$-\alpha_1$	$-\alpha_3$	α_1			
$SU(3)$	α_1	$-\alpha_1$	$-\alpha_1$	α_1	0	$m_1^{(2)}\alpha_1 + m_2^{(2)}\alpha_2$	0
	α_2	$\sum_i^2 \alpha_i$	$-\alpha_2$	$-\sum_i^2 \alpha_i$			
$SU(2)$	e_1	$-e_1$	$-e_1$	e_1	$m^{(1)}e_1$	$m_1^{(2)}e_1$	

Table 1: Building blocks for $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds.

4. Chiral spectra for standard embedding

In $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds a chiral spectrum originates from twisted sectors. Hence, a modification of the fixed point structure affects the chiral matter content of the model.

For example, if the orbifold action is standard embedded into the $\mathrm{E}_8 \times \mathrm{E}_8$ gauge group of the ten dimensional heterotic string the unbroken gauge group in four dimensions contains an E_6 factor [15]. The untwisted sector gives rise to three chiral multiplets in the **27** and three chiral multiplets in the $\overline{\mathbf{27}}$ of E_6, whereas the twisted sectors provide different numbers of **27**'s and $\overline{\mathbf{27}}$'s.

As in [19] we could obtain the chiral spectrum by explicit construction. There is, however, a shortcut employing modular invariance of the partition function (see e.g. the third reference in [2] and [22]). In [19] this method is briefly described in the context of non factorisable $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds. In the following we repeat that discussion giving also more details. In the end, we obtain a closed formula for the number of generations (**27**'s) and anti-generations ($\overline{\mathbf{27}}$'s) for the standard embedded orbifold. (For embeddings other than the standard embedding and in the presence of Wilson lines the same method works, the $\mathrm{E}_8 \times \mathrm{E}_8$ sector of the partition function has to be considered explicitly, though.)

We write the partition function as

$$Z(\tau, \overline{\tau}) = \sum_{i,j} \Delta_{i,j} Z_{i,j}, \tag{4.1}$$

where i, j label the elements of the orbifold group, for example

$$Z_{\theta_1,\theta_2} = \frac{1}{4} \underset{\theta_1 \text{ twisted}}{\mathrm{Tr}} \theta_2 q^{L_0} \overline{q}^{\overline{L_0}}, \tag{4.2}$$

with $q = \exp(2\pi i \tau)$. The factors $\Delta_{i,j}$ are fixed by modular invariance. In the untwisted sector they are the same as in the uncompactified case (e.g. one for bosons). The number of generations can be determined by identifying the contributions from massless chiral fermions to Z.

In the θ_1 twisted sector, for example, terms with an identity or θ_1 insertion in the trace have identical contributions from GSO even chiral and anti-chiral fermions. For terms with a θ_2 or $\theta_1\theta_2$ insertion, GSO even chiral and anti-chiral fermions contribute with opposite

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signs.[4] We find for the number of chiral fermions, M, and anti-chiral fermions, N, from the θ_1 twisted sector

$$M = \tilde{\Delta}_{\theta_1,1} + \tilde{\Delta}_{\theta_1,\theta_1} + \tilde{\Delta}_{\theta_1,\theta_2} + \tilde{\Delta}_{\theta_1,\theta_1\theta_2}, \tag{4.3}$$

$$N = \tilde{\Delta}_{\theta_1,1} + \tilde{\Delta}_{\theta_1,\theta_1} - \tilde{\Delta}_{\theta_1,\theta_2} - \tilde{\Delta}_{\theta_1,\theta_1\theta_2}, \tag{4.4}$$

where the tilde indicates absorption of numerical factors coming from the trace. For the factorisable case we know that there are 16 generations and zero anti-generations. Therefore we must have

$$\tilde{\Delta}_{\theta_1,1} + \tilde{\Delta}_{\theta_1,\theta_1} = \tilde{\Delta}_{\theta_1,\theta_2} + \tilde{\Delta}_{\theta_1,\theta_1\theta_2} = 8 \tag{4.5}$$

in the factorisable case.

What we will do now is to identify dependencies on the compactification lattice in the $\tilde{\Delta}$'s and in turn fix their value by our knowledge of the factorisable case. For example, in the contribution from untwisted states with a θ_1 insertion in the trace there will be a sum over winding and momenta corresponding to the two compact directions which are not affected by θ_1. The same happens in the θ_1 twisted sector if an identity or θ_1 is inserted into the trace. A modular transformation takes the untwisted sector sums to the twisted sector ones which can be seen by Poisson resumming winding and momenta.

For non factorisable compactifications there are two types of two-tori appearing. One comes from the θ_1 projection of the compactification lattice. (This is not necessarily a sublattice.) The other one is spanned by an invariant sublattice of the compactification lattice. For the contribution from the untwisted sector with θ_1 insertion, windings are summed over the invariant lattice whereas momenta are summed over the invariant dual lattice (only these contribute to the trace). A subtlety for non factorisable compactifications is that the invariant dual lattice is not the dual of the invariant lattice but the dual of the projected lattice as can be seen from the defining property

$$\delta_{ij} = (e_i^\star, e_j) = \left(\frac{1+\theta_1^T}{2} e_i^\star\, ,\, e_j\right) = \left(e_i^\star\, ,\, \frac{1+\theta_1}{2} e_j\right). \tag{4.6}$$

For the twisted sector windings are on the projected lattice (strings do have to close only up to θ_1 identifications). Momenta are quantised w.r.t. to the dual of the invariant sublattice which spans the fixed torus.

After Poisson resumming the untwisted sector the erstwhile momentum sum becomes a winding sum over the projected lattice whereas the erstwhile winding sum becomes a momentum sum over the dual of the invariant lattice. This is consistent with modular invariance. The Poisson resummations provide also a numerical factor of the ratio[5] between the volume of the projected lattice ($\mathrm{vol}(\Lambda^\perp)$) and the volume of the invariant lattice ($\mathrm{vol}(\Lambda^{\mathrm{inv}})$). Hence we conclude

$$\tilde{\Delta}_{\theta_1,1} + \tilde{\Delta}_{\theta_1,\theta_1} = 8\frac{\mathrm{vol}\left(\Lambda^\perp\right)}{\mathrm{vol}\left(\Lambda^{\mathrm{inv}}\right)}, \tag{4.7}$$

[4] The corresponding Δ's seem not to be fixed by modular invariance since they multiply zero. This will, however, not be important for the discussion below. Changing their sign is called discrete torsion which for $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds just swaps the number of families with the number of anti-families [15, 23].

[5] The volume of a dual lattice is inverse to the volume of the lattice.

where we used that for the known factorisable case the ratio of the volumes is one.

In the traces over the θ_1 twisted sector with a θ_2 or $\theta_1\theta_2$ insertion there will be no sum over windings or momenta since these quantum numbers are not invariant under the inserted operator if they differ from zero. There is, however, also a dependence on the compactification lattice. Only if the position of a θ_1 twisted state is invariant under the inserted operator that state will contribute to the trace. Calling the number of points, which are invariant under θ_i and θ_j, $\chi(\theta_i, \theta_j)$ and comparing to the factorisable case (in which these numbers are 64) we find

$$\tilde{\Delta}_{\theta_1,\theta_2} + \tilde{\Delta}_{\theta_1,\theta_1\theta_2} = \frac{\chi(\theta_1, \theta_2)}{8}, \tag{4.8}$$

where we used the obvious relation $\chi(\theta_1, \theta_2) = \chi(\theta_1, \theta_1\theta_2)$.

Collecting everything we obtain the following relations for the number of families M and anti-families N from imposing modular invariance

$$M = 8\frac{\mathrm{vol}\left(\Lambda^{\perp}\right)}{\mathrm{vol}\left(\Lambda^{\mathrm{inv}}\right)} + \frac{\chi(\theta_1, \theta_2)}{8}, \tag{4.9}$$

$$N = 8\frac{\mathrm{vol}\left(\Lambda^{\perp}\right)}{\mathrm{vol}\left(\Lambda^{\mathrm{inv}}\right)} - \frac{\chi(\theta_1, \theta_2)}{8}. \tag{4.10}$$

We added the resulting numbers for families and anti-families to the list of models in the appendix.

5. Further phenomenological aspects

5.1 Wilson line

Twisted sector states corresponding to the same orbifold group element but to a different fixed point or torus form identical four dimensional spectra. Wilson lines can lift that degeneracy. Thus, they provide an important tool for the reduction of the number of families [2, 24, 25]. Here, we study discrete Wilson lines possible in 3D SU(4), 2D SU(3) and 1D SU(2) $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds.

A Wilson line can be viewed as a non trivial embedding of a shift by a lattice vector into the gauge group. In the bosonic formulation of the heterotic $E_8 \times E_8$ theory, there are 16 left moving bosons compactified on an $E_8 \times E_8$ root lattice. Using that language, we embed the space group element (θ, V) as a shift $(V_\theta + W_V)$ into the 16 left moving directions.

The shift vector V of 6D compact space is embedded as the shift W_V into the $E_8 \times E_8$ lattice, and W_V is called Wilson line along the cycle V. Different fixed points (tori) in the same twisted sector are fixed under the action of space group elements with different shifts in the 6D space. Thus, Wilson lines can lift the degeneracy of the spectrum within the same twisted sector. Since V and θV are identical on the orbifold, the corresponding Wilson lines should be equivalent, i.e. differ at most by an $\Lambda_{E_8\times E_8}$ lattice vector.

orbifold	twists	conditions for Wilson lines (up to $\Lambda_{E_8\times E_8}$)
$SU(4)$	eq. (3.7)	$2W_{\alpha_1}=2W_{\alpha_2}=0, \quad W_{\alpha_1}=W_{\alpha_3}$
$SU(3)$	eq. (3.12)	$W_{\alpha_1}=0, \quad 2W_{\alpha_2}=0$
$SU(2)$	eq. (3.19)	$2W_{\alpha_1}=0$

Table 2: Conditions for Wilson lines.

As an example we consider the 3D SU(4) $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold with the twist (3.7). The twist θ_1 requires that $W_{\alpha_1}+W_{\alpha_3}$ and $2W_{\alpha_2}$ should be on $\Lambda_{E_8\times E_8}$, where W_{α_i} denotes the Wilson line along α_i. Furthermore, the twist θ_2 requires $2W_{\alpha_1}$ and $2W_{\alpha_3}$ to be on $\Lambda_{E_8\times E_8}$. As a result, we obtain the condition for Wilson lines as

$$2W_{\alpha_1}=2W_{\alpha_2}=0, \qquad W_{\alpha_1}=W_{\alpha_3}, \tag{5.1}$$

up to $\Lambda_{E_8\times E_8}$.

Similarly, we can study conditions on Wilson lines for other building blocks, that is, SU(3) and SU(2) $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifolds. Results are shown in table 2. In addition to these geometrical consistency conditions one has to impose modular invariance conditions [2] when including Wilson lines in the full heterotic construction.

5.2 Yukawa couplings

Here we consider Yukawa couplings in non-factorisable $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold models. We focus on the space group selection rule for allowed Yukawa couplings, because selection rules for other parts are the same as in conventional factorisable models. In particular, we study the 3-point coupling of three twisted states corresponding to the space group elements, (θ_1,V_1), (θ_2,V_2) and (θ_3,V_3). Their coupling is allowed if the product of three space group elements is equal to identity. Note that the space group element (θ_i,V_i) is equivalent to $(\theta_i,V_i+(1-\theta_i)\Lambda)$, up to conjugacy class. Therefore, the condition for allowed coupling is obtained as [7, 25]

$$(\theta_1,V_1)(\theta_2,V_2)(\theta_3,V_3)=(1,0), \tag{5.2}$$

up to $(1-\theta_i)\Lambda$.

As an example, let us consider the SU(4) $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold. Then, we study the 3-point coupling corresponding to three twisted states, $(\theta_1,m^{(1)}\alpha_2)$, $(\theta_2,m^{(2)}\alpha_1)$ and $(\theta_3,m^{(3)}(\alpha_1+\alpha_2))$. Note that $\cup_i(1-\theta_i)\Lambda_{SU(4)}$ is spanned by $2\alpha_1$ and $2\alpha_2$ as well as $\alpha_1-\alpha_3$. Thus, this coupling is allowed when

$$m^{(1)}+m^{(3)}= \text{ even}, \qquad m^{(2)}+m^{(3)}= \text{ even}. \tag{5.3}$$

For the other orbifolds, we can obtain conditions on allowed couplings in a similar way. Results are shown in table 3. The fifth column shows conditions for allowed $T_1T_2T_3$ couplings. Note that only $T_1T_2T_3$ type of couplings are allowed, while the other types, e.g. $T_iT_iT_j$ $(i\neq j)$ and $T_iT_iT_i$ are forbidden by the point group selection rule. Only diagonal couplings are allowed on 3D SU(4) and 1D SU(2) $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifolds, that is, when we chose two states, the other state to be allowed to couple is uniquely fixed. However, off-diagonal

orbifold	T_1	T_2	T_3	Conditions
$SU(4)$	$(\theta_1, m^{(1)}\alpha_2)$	$(\theta_2, m^{(2)}\alpha_1)$	$(\theta_3, m^{(3)}(\alpha_1+\alpha_2))$	$m^{(1)} + m^{(3)} =$ even $m^{(2)} + m^{(3)} =$ even
$SU(3)$	$(\theta_1, 0)$	$(\theta_2, m_1^{(2)}\alpha_1 + m_2^{(2)}\alpha_2)$	$(\theta_3, 0)$	$m_2^{(2)} = 0$
$SU(2)$	$(\theta_1, m^{(1)}e_1)$	$(\theta_2, m_1^{(2)}e_1)$		$m^{(1)} + m^{(2)} =$ even

Table 3: Conditions for allowed $T_1T_2T_3$ couplings.

couplings are allowed on 2D SU(3)/$\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds. On the 2D SU(3)/$\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, both states corresponding to $(\theta_2, m_1^{(2)}\alpha_1)$ for $m_1^{(2)} = 0, 1$ in the T_2 sector can couple with the T_1 and T_3 sectors corresponding to $(\theta_1, 0)$ and $(\theta_3, 0)$.

Here, we note an (exclusion) relation between the possibility of having off-diagonal couplings and discrete Wilson lines. For example, the 2D SU(3) $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold does not allow non-trivial Wilson lines along α_1, i.e. $W_{\alpha_1} = 0$ up to $\Lambda_{E_8\times E_8}$. Unfortunately, it is not clear how discrete Wilson lines contribute to magnitudes and CP phases of Yukawa couplings. However, such ignorance does not affect two types of couplings, $(\theta_1, 0)(\theta_2, 0)(\theta_3, 0)$ and $(\theta_1, 0)(\theta_2, \alpha_1)(\theta_3, 0)$, because there is no non-trivial discrete Wilson line $A_{\alpha_1} = 0$. Thus, when off-diagonal couplings are allowed, there is no non-trivial discrete Wilson lines to distinguish them. This aspect has been found, previously, in $\mathbb{Z}_N$ orbifold models [10].

Magnitudes of allowed Yukawa couplings can be computed by the usual method [6−9]. In these models, all of allowed Yukawa couplings are of $O(1)$ w.r.t. the string scale up to field redefinitions needed for obtaining canonical Kähler potentials.

6. Conclusion and discussion

In the present paper, we classified $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds of non-factorisable six-tori. We restricted our attention to orbifolds that lead, if taken as the compact space for heterotic strings, to $N = 1$ supersymmetry in four dimensions. We found that all topologically inequivalent orbifolds can be composed from the following building blocks: 3D SU(4), 2D SU(3) and 1D SU(2) $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds. Using them, we have classified eight 6D $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds, including the conventional factorisable one. In the factorisable $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models, each of the three twisted sectors has 16 fixed tori, whereas for non-factorisable orbifold models these numbers are smaller. In the model with the minimal numbers, each of three twisted sectors has four fixed tori. Thus, these models have a variety of generation numbers. For example, in the standard embedding, the smallest number of net generations among these eight classes is equal to six, while the largest number is 48, which is obtained in the conventional factorisable model. We have also studied discrete Wilson lines and selection rules for allowed Yukawa couplings. These features are different from the conventional factorisable $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models.

The non-factorisable models allow off-diagonal couplings, while the factorisable one allows only diagonal couplings. However, it seems difficult to realise quark/lepton masses and mixing angles by use of only 3-point couplings in non-factorisable $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models with the minimal number of Higgs fields.

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In order to obtain realistic fermion mass matrices, it would be interesting to introduce more Higgs fields and/or use higher dimensional operators. The selection rules for generic n-point couplings can be obtained by extending our analysis of the 3-point couplings. Furthermore, it is important to study which type of non-Abelian flavour symmetries can appear from non-factorisable $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models [3, 26].

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A. Eight classes of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds

In this appendix, we list 8 classes of models. Before giving the details of each model in subsections A.1,–A.8, we summarise some of their properties in table 4. The Euler number, χ, is computed according to the general formula [1]

$$\chi = \frac{1}{|G|} \sum_{[g,h]=0} \chi(g,h)\,, \tag{A.1}$$

where $|G|$ is the order of the orbifold group with elements g, h and $\chi(g,h)$ is the number of points which are simultaneously fixed under the action of g and h. For the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold there are six contributing pairs of non trivial orbifold group elements, each pair leaves the same points invariant. The order of the orbifold group is 4, and hence we can simplify (A.1) to

$$\chi = \frac{3}{2}\chi(\theta_1, \theta_2)\,. \tag{A.2}$$

For the standard embedded orbifold the net number of generations should equal $\chi/2$. Following section 4, we can also compute the number of families, M_i, (anti-families, N_i) from each sector, i.e. $i = 0, \ldots, 3$, where $i = 0$ corresponds to the untwisted sector. We summarise the results in table 4.

In the following subsections we list the details for each model.

A.1 $\mathrm{SU}(2)^6$

We combine six SU(2) $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds to construct 6D $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold as follows,

		e_1	e_2	e_3	e_4	e_5	e_6	N_{fp}
θ_1	:	$-$	$-$	$-$	$-$	$+$	$+$	16
θ_2	:	$+$	$+$	$-$	$-$	$-$	$-$	16
θ_3	:	$-$	$-$	$+$	$+$	$-$	$-$	16

where e_i denotes the simple root of the i-th SU(2) root lattice, and $+$ and $-$ denote how e_i transforms under the twist θ_i. The number of fixed points (tori) for each twist is denoted by N_{fp}. At any rate, this is the conventional factorisable 6D $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold.

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Model	χ	(M_0, N_0)	(M_1, N_1)	(M_2, N_2)	(M_3, N_3)	$\sum_{i=0}^{3} (M_i, N_i)$
A.1	96	(3,3)	(16,0)	(16,0)	(16,0)	(51,3)
A.2	48	(3,3)	(12,4)	(8,0)	(8,0)	(31,7)
A.3	24	(3,3)	(10,6)	(4,0)	(4,0)	(21,9)
A.4	48	(3,3)	(8,0)	(8,0)	(8,0)	(27,3)
A.5	24	(3,3)	(6,2)	(6,2)	(4,0)	(19,7)
A.6	24	(3,3)	(6,2)	(4,0)	(4,0)	(17,5)
A.7	24	(3,3)	(4,0)	(4,0)	(4,0)	(15,3)
A.8	12	(3,3)	(3,1)	(3,1)	(3,1)	(12,6)

Table 4: The eight models for standard embedding: The numbers M_i (N_i) denote the number of (anti-)families, where the index i labels the twist sector ($i = 0$ means untwisted). The net number of families is given by half the Euler number, χ.

A.2 SU(3)×SU(2)^4

We combine SU(3) $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold and four SU(2) $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds as follows,

	SU(3)		e_3	e_5	e_5	e_6	N_{fp}
θ_1 :	$-$	$-$	$-$	$-$	$+$	$+$	16
θ_2 :	$+$	$-$	$+$	$-$	$-$	$-$	8
θ_3 :	$-$	$+$	$-$	$+$	$-$	$-$	8

where e_i denotes the simple root of the i-th SU(2) root lattice. The twist θ_1 is 2D $\mathbb{Z}_2$ rotation on the SU(3) root lattice, and the twist θ_3 the Weyl reflection corresponding to one of SU(3) simple roots, α_1.

A.3 SU(3)^2×SU(2)^2-I

We combine two SU(3) $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds and two SU(2) $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds as follows,

	SU(3)		SU(3)		e_5	e_6	N_{fp}
θ_1 :	$-$	$-$	$-$	$-$	$+$	$+$	16
θ_2 :	$+$	$-$	$+$	$-$	$-$	$-$	4
θ_3 :	$-$	$+$	$-$	$+$	$-$	$-$	4

where the twist θ_1 is 2D $\mathbb{Z}_2$ rotation on both the first and second SU(3) lattices. The twist θ_3 is the Weyl reflection corresponding to the first simple roots, i.e. α_1 and α_3, for both the first and second SU(3) root lattices.

A.4 SU(4)×SU(2)^3

We combine SU(4) $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold and three SU(2) $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds as follows,

	SU(4)			e_4	e_5	e_6	N_{fp}
θ_1 :	$-$	$-$	$+$	$-$	$-$	$+$	8
θ_2 :	$+$	$-$	$-$	$+$	$-$	$-$	8
θ_3 :	$-$	$+$	$-$	$-$	$+$	$-$	8

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where the twist θ_1 is a product of the outer automorphism replacing between α_1 and α_3 and the total $\mathbb{Z}_2$ rotation, $\alpha_i \to -\alpha_i$ $(i = 1, 2, 3)$, for the SU(4) lattice, while the twist θ_2 is a sum of two Weyl reflections for α_1 and α_3.

A.5 $\mathbf{SU(3)^2 \times SU(2)^2}$-II

We combine two SU(3) $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds and two SU(2) $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds as follows,

	SU(3)		SU(3)		e_5	e_6	N_{fp}
θ_1 :	$-$	$-$	$-$	$+$	$-$	$+$	8
θ_2 :	$+$	$-$	$-$	$-$	$+$	$-$	8
θ_3 :	$-$	$+$	$+$	$-$	$-$	$-$	4

where the twist θ_1 is the 2D $\mathbb{Z}_2$ rotation on the first SU(3) lattice and the twist θ_3 is the Weyl reflection corresponding to one of simple roots of the first SU(3) lattice, α_1. Moreover, the twist θ_1 is the Weyl reflection corresponding to one of simple roots of the second SU(3) lattice, α_3, and the twist θ_2 is the 2D $\mathbb{Z}_2$ rotation on the second SU(3) lattice.

A.6 $\mathbf{SU(4) \times SU(3) \times SU(2)}$

We combine SU(4) $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, SU(3) $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold and SU(2) $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold as follows,

	SU(4)			SU(3)		e_6	N_{fp}
θ_1 :	$-$	$-$	$+$	$-$	$-$	$+$	8
θ_2 :	$+$	$-$	$-$	$+$	$-$	$-$	4
θ_3 :	$-$	$+$	$-$	$-$	$+$	$-$	4

where the twist θ_1 is a product of the outer automorphism replacing between α_1 and α_3 and the total $\mathbb{Z}_2$ rotation, $\alpha_i \to -\alpha_i$ $(i = 1, 2, 3)$, for the SU(4) lattice, while the twist θ_2 is a sum of two Weyl reflections for α_1 and α_3. In addition, the twist θ_1 is the 2D $\mathbb{Z}_2$ rotation on the SU(3) lattice, and the twist θ_3 is the Weyl reflection corresponding to one of SU(3) simple roots, α_4.

A.7 $\mathbf{SU(4)^2}$

We combine two SU(4) $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds as follows,

	SU(4)			SU(4)			N_{fp}
θ_1 :	$-$	$-$	$+$	$-$	$-$	$+$	4
θ_2 :	$+$	$-$	$-$	$+$	$-$	$-$	4
θ_3 :	$-$	$+$	$-$	$-$	$+$	$-$	4

where the twist θ_1 is a product of the outer automorphism replacing between α_1 and α_3 (α_4 and α_6) and the total $\mathbb{Z}_2$ rotation, $\alpha_i \to -\alpha_i$, for the first (second) SU(4) lattice. In addition, the twist θ_2 is a sum of two Weyl reflections for α_1 and α_3 (α_4 and α_6) on the first (second) SU(4) lattice.

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A.8 $\mathrm{SU}(3)^3$

We combine three SU(3) $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds as follows,

	SU(3)		SU(3)		SU(3)		N_{fp}
θ_1 :	$-$	$-$	$-$	$+$	$-$	$+$	4
θ_2 :	$+$	$-$	$+$	$-$	$-$	$-$	4
θ_3 :	$-$	$+$	$-$	$-$	$+$	$-$	4

where the twist θ_1 is the 2D $\mathbb{Z}_2$ rotation on the first SU(3) root lattice and is the Weyl reflection for one of simple roots for both the second and third SU(3) root lattices. The twist θ_3 is the Weyl reflection for one of simple roots for the first SU(3) root lattice and is the 2D $\mathbb{Z}_2$ rotation on the second SU(3) root lattice. Furthermore, the twist θ_2 is the Weyl reflection for one of simple roots on the third SU(3) root lattice.

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